

Lifted Inference for Relational Continuous Models

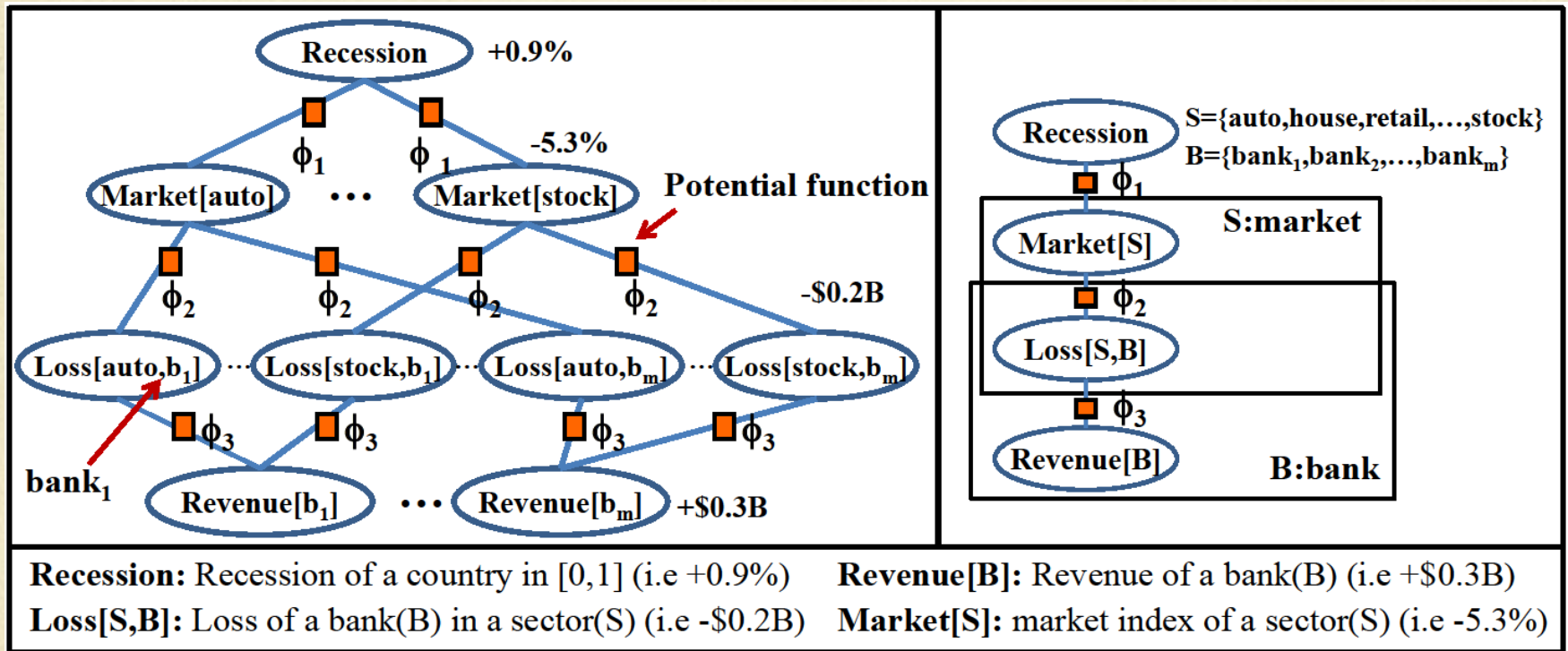
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What are Relational Continuous Models (RCMs)?

- An example RCM



Why we use relational models?

They allow compact representations and efficient inference possible (in some cases).

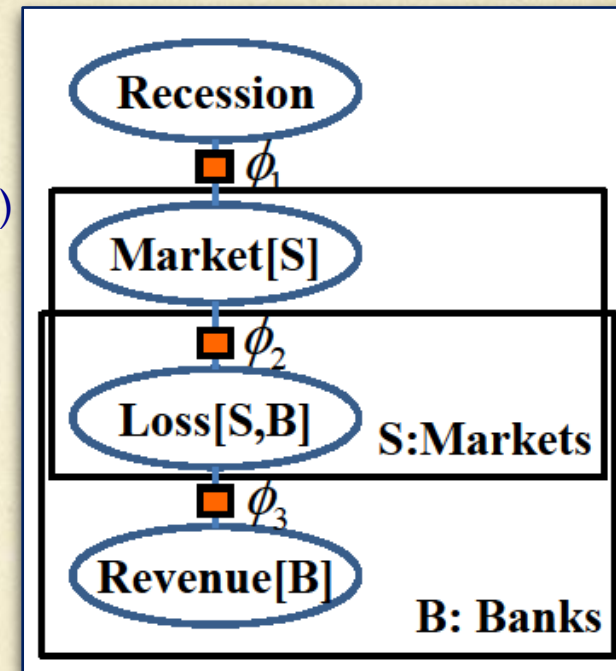
Contents

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 - Notations: Relational Continuous Models
 - Problem definition: an inference problem
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Notations: Relational Continuous Models (RCMs)

○ Notations in RCMs

- A set of parfactors
 - A parfactor $g: (L, C, R, \phi)$
 - L: a set of logical variables (e.g. {S:Market, B:Banks})
 - C: constraints (e.g. Market \neq stock)
 - R: a set of relational atoms (e.g. Market[S])
 - ϕ : a potential (e.g. $\phi_1(\text{Recession}, \text{Market}[S])$)
- There are three parfactors
 - $(\{S\}, \{\}, \{\text{Recession}, \text{Market}[S]\}, \phi_1)$
 - $(\{S, B\}, \{\}, \{\text{Market}[S], \text{Loss}[S, B]\}, \phi_2)$
 - $(\{B\}, \{\}, \{\text{Loss}[S, B], \text{Revenue}[B]\}, \phi_3)$
- Probability density of an instance

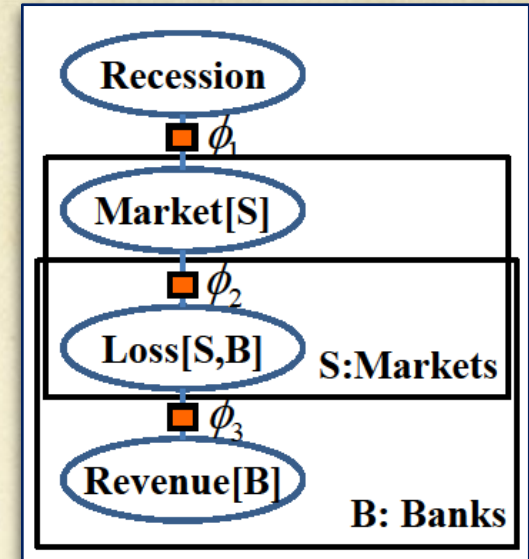


$$\prod_{s \in S} \phi_1(\text{Recession}, \text{Market}(s)) \cdot \prod_{s \in S, b \in B} \phi_2(\text{Market}(s), \text{Loss}(s, b)) \cdot \prod_{s \in S, b \in B} \phi_3(\text{Loss}(s, b), \text{Revenue}(b))$$

Problem Definition: an inference problem

- Given
 - A RCM
 - Observations
(e.g. Market(auto) = -5.3%, Revenue(Pacific Bank)=-\$0.2B)
 - Query random variables
(e.g. Recession)
- Output
 - The marginal probability density of the query variables.
- Conditions
 - The potentials in the RCM are arity 2 with the following forms

$$\phi_{RN}(X[\dots], Y[\dots]) = c \cdot \exp\left(-\frac{(X[\dots] - Y[\dots])^2}{\sigma^2}\right)$$



Relational Normal Potentials

- Pair-wise Relational Normals (RNs)
 - A family of Gaussian potentials with following forms
 - The potential is defined over two relational atoms ($X[\dots]$, $Y[\dots]$)

$$\phi_{RN}(X[\dots], Y[\dots]) = c \cdot \exp\left(-\frac{(X[\dots] - Y[\dots])^2}{\sigma^2}\right)$$

$$\phi^{RN}(Markov[2], Color[2, B]) = c \cdot \exp\left(-\frac{(Markov[2] - Color[2, B])^2}{\sigma^2}\right)$$

- Pair-wise Relational Normals with Means (RNMs)

$$\phi_{RNM}(X[\dots], Y[\dots]) = c \cdot \exp\left(-\frac{(X[\dots] - Y[\dots] - \mu)^2}{\sigma^2}\right)$$

Eliminating random variables in Gaussian Potentials

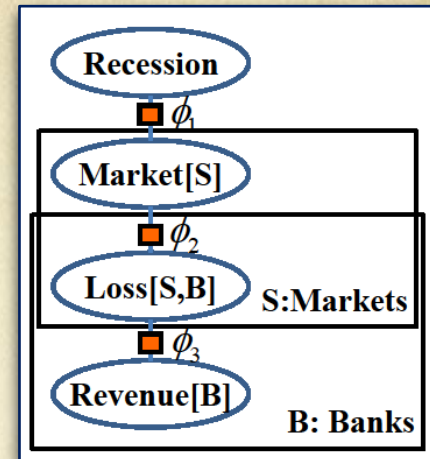
- Gaussian Potentials are quadratic exponential family.
 - For all random variables, it is a quadratic form (e.g. $R(b_i)$).

$$\exp(-aR(b_i)^2 + bR(b_i) + c)$$

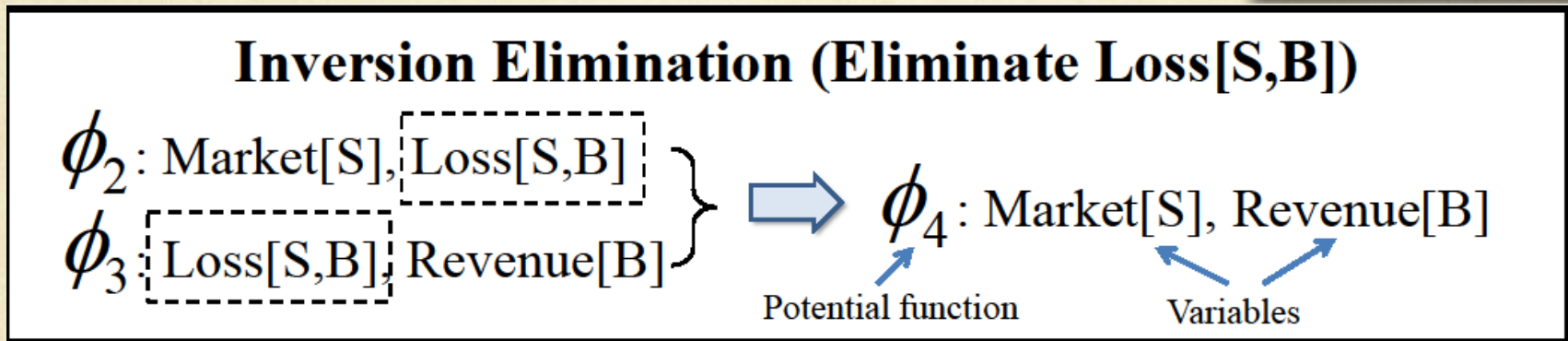
- a is a positive constant. b and c are terms without $R(b_i)$.
- Marginalizing with respect to a random variables, $R(b_i)$

$$\int_{R(b_i)} \exp(-aR(b_i)^2 + bR(b_i) + c) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c\right)$$

Inversion Elimination



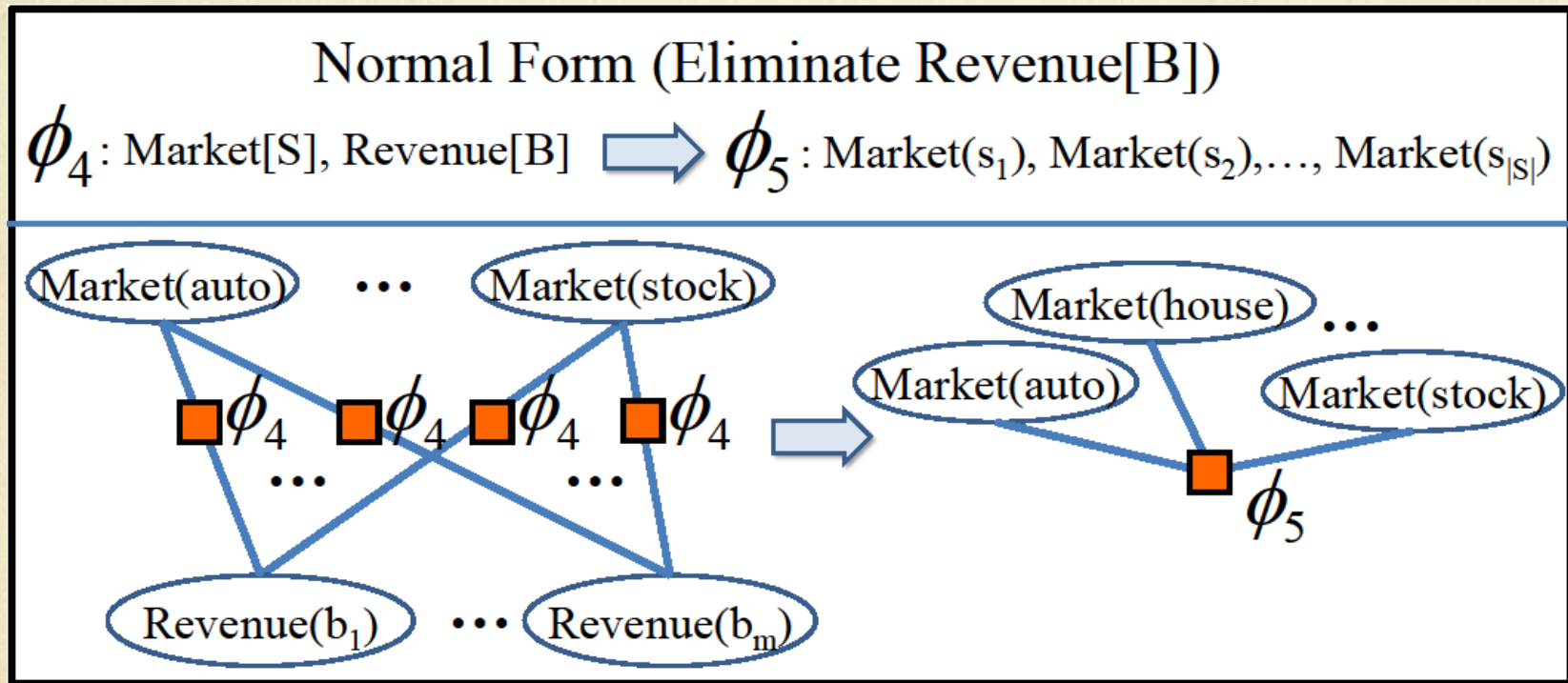
- Eliminating Loss[S,B]



$$\begin{aligned}
 & \int_{Loss[S,B]} \phi_2(Market[S], Loss[S,B]) \cdot \phi_3(Loss[S,B], Revenue[B]) \\
 &= c \cdot \int_{Loss[S,B]} \exp\left(-\frac{(Market[S] - Loss[S,B])^2}{\sigma_1^2} - \frac{(Loss[S,B] - Revenue[B])^2}{\sigma_2^2}\right) \\
 &= c' \cdot \exp\left(-\frac{(Market[S] - Revenue[B])^2}{(\sigma_1^2 + \sigma_2^2)}\right) \\
 &= \phi_4(Market[S], Revenue[B])
 \end{aligned}$$

Eliminating random variables in a ground level

- Eliminating Revenue(b_1), ..., Revenue(b_m)



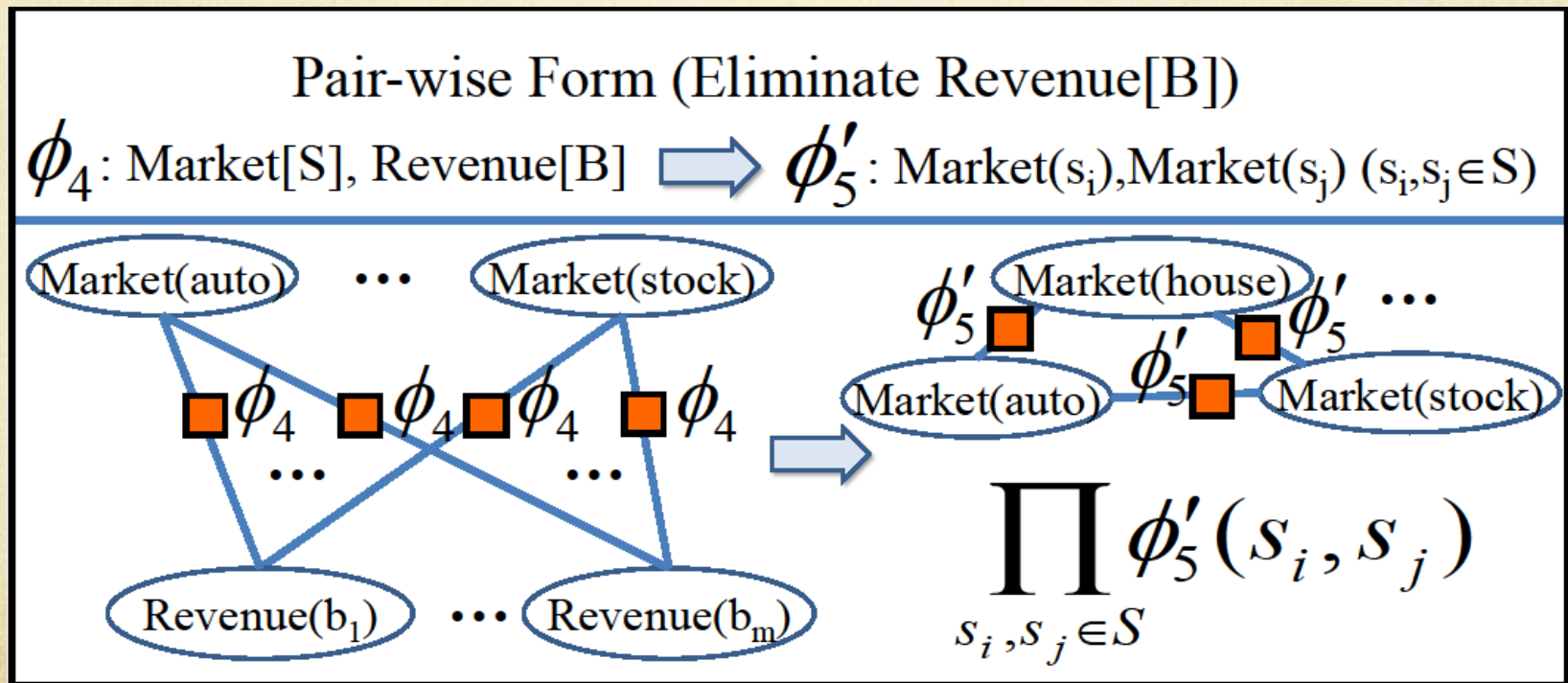
Eliminating random variables in a ground level

○ Details of marginalization

$$\begin{aligned}
 \int_{R(b_i)} \prod_{s \in S} \phi_4(M(s), R(b_i)) dR(b_i) &= \int_{R(b_i)} \exp\left(\sum_{s \in S} -\frac{(R(b_i) - M(s))^2}{\sigma^2}\right) \\
 &= \int_{R(b_i)} -\frac{|S|}{\sigma^2} R(b_i)^2 + \frac{2\left(\sum_{s \in S} M(s)\right)}{\sigma^2} R(b_i) - \frac{\left(\sum_{s \in S} M(s)^2\right)}{\sigma^2} \\
 &= c \cdot \exp\left(\frac{\left(\sum_{s \in S} M(s)\right)^2}{2\sigma^2 \cdot |S|} - \frac{\sum_{s \in S} M(s)^2}{2\sigma^2}\right) \\
 &= \phi_5(M(s_1), M(s_2), \dots, M(s_{|S|}))
 \end{aligned}$$

Our method – Eliminating random variables in a lifted level with the Pair-wise form

- We can make the pair-wise form after eliminating a random variable in the Pair-wise Relational Normals.



Our method – Eliminating random variables in a lifted level with the Pair-wise form

○ Details of marginalization

$$\begin{aligned}
 \int_{R(b_i)} \prod_{s \in S} \phi_4(M(s), R(b_i)) dR(b_i) &= \int_{R(b_i)} \exp\left(\sum_{s \in S} -\frac{(R(b_i) - M(s))^2}{\sigma^2}\right) \\
 &= \int_{R(b_i)} -\frac{|S|}{\sigma^2} R(b_i)^2 + \frac{2\left(\sum_{s \in S} M(s)\right)}{\sigma^2} R(b_i) - \frac{\left(\sum_{s \in S} M(s)^2\right)}{\sigma^2} \\
 &= c \cdot \exp\left(\frac{\left(\sum_{s \in S} M(s)\right)^2}{2\sigma^2 \cdot |S|} - \frac{\sum_{s \in S} M(s)^2}{2\sigma^2}\right) \\
 &= c \cdot \prod_{1 \leq i < j \leq |S|} \exp\left(-\frac{(M(s_i) - M(s_j))^2}{2\sigma^2 \cdot |S|}\right) \\
 &= c \cdot \prod_{1 \leq i < j \leq |S|} \phi_5'(M(s_i), M(s_j))
 \end{aligned}$$

Our method – Eliminating random variables in a lifted level with the Pair-wise form

- Generalization
 - We made some algebraic notations

$$X_{[m]} = \sum_{1 \leq i \leq m} x_i \quad \text{e.g.} \quad M_{[S]} = \sum M(s)$$

$$X_{[m]^2} = \sum_{1 \leq i \leq m} x_i^2 \quad \text{e.g.} \quad M_{[S]^2} = \sum_{s \in S} M(s)^2$$

$$X_{[m][m]} = \sum_{1 \leq i \leq m} x_i \cdot x_j$$

$$\left(X_{[m]^2} \right)^2 = X_{[m]^2} + 2X_{[m][m]}$$

Computational Complexity

- Given
 - $|U|$ is the all random variables
 - $|N|$ is the relational atoms
 - Normally, $|U| \gg |N|$
- Marginalization of n random variables in a ground level

$$O(n \cdot |U|^2)$$

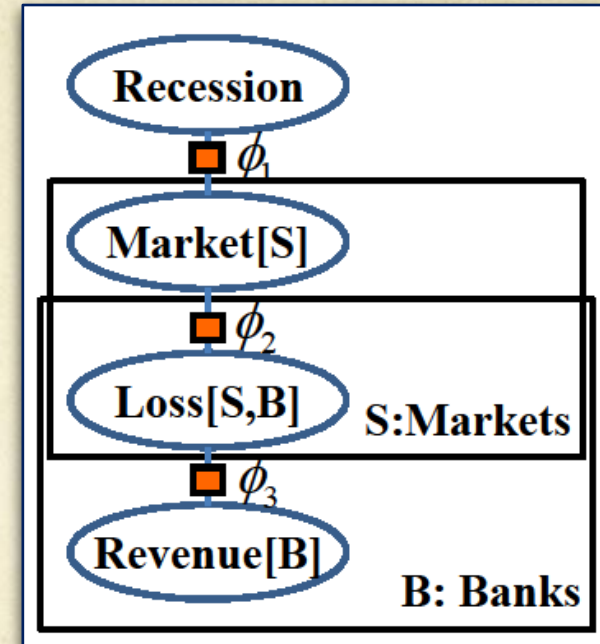
- Marginalization of n random variables with our algorithm

$$O(n \cdot |N|^2)$$

Experiments

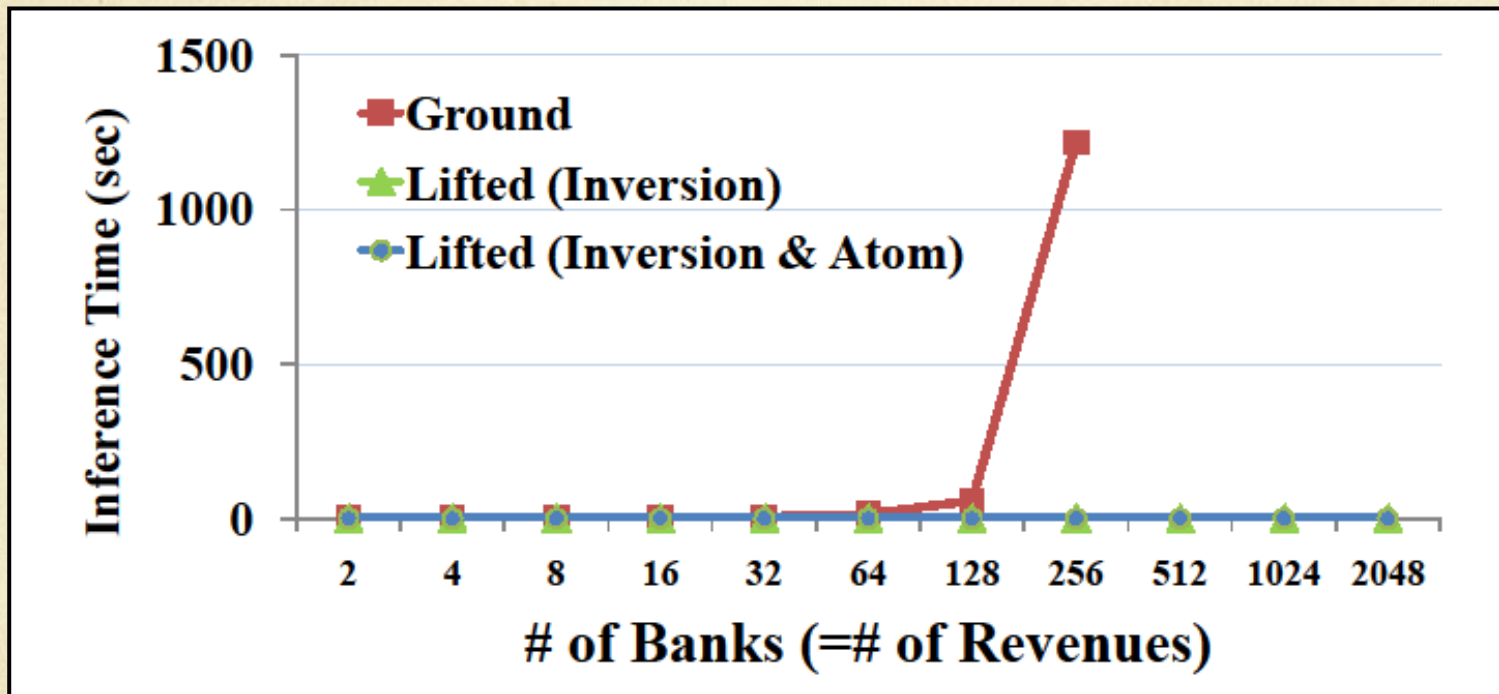
- Experimental Settings
 - Given the econometric example
 - One observation for a market variable
 - One observation for a revenue variable
 - Output: Recession variable
 - Compare three algorithms
 - A ground inference
 - A lifted inference with only inversion
 - A lifted inference with inversion and pair-wise

- Exp 1: Increase the number of banks ($|B|$)
- Exp 2: Increase the number of market sectors ($|S|$)



Increase the number of Banks

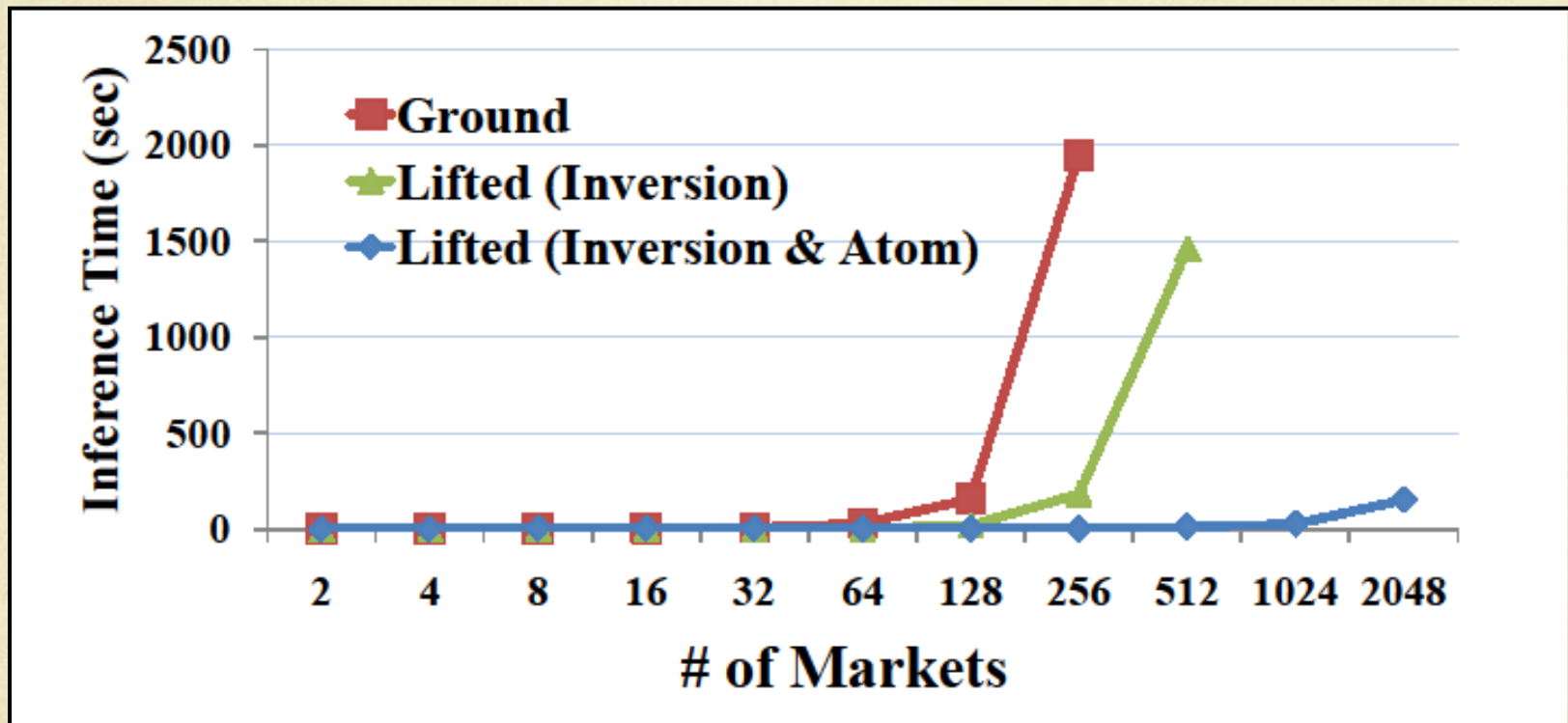
- Time for exact inference



- When # of banks is $2n$, # of random variables is $4n + |\text{Market}| + 1$.
- The computational complexity of ground algorithm is quadratic $16n^3$. 16

Increase the number of Markets

- Time for exact inference



Additional Details in the Paper

- The paper includes
 - A condition which makes the relational normal is a probability density function.
 - A proof that the product of potentials is a probability density function given the condition.
 - Three conditions for exact lifted inference for general potentials (including non-Gaussian)
 - Analytically integrable
 - Closed under product operations
 - Closed under marginalization, thus represented with the product of relational pairwise form

Conclusion and Future work

- What we have seen so far
 - A linear time exact inference algorithm for continuous domains with Gaussian potentials.
 - The lifted algorithm substantially reduces the inference time compared to the ground-based algorithm.
- Future work
 - An algorithm for other potential (Gamma, Beta, Dirichlet ...) would be promising.
 - There are many applications with Hybrid domains.