Lifted Relational Kalman Filtering

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* speaker

Kalman Filter

- The Kalman Filter (KF) recursively estimates the state variables of a dynamic system. It consists of three components:
  1. A joint distribution of state variables: multivariate Gaussian.

$$P(X_t) \propto \exp(-(X_t - \mu)\Sigma^{-1}(X_t - \mu)^T)$$

  2. Transition models: (multivariate) Gaussian
  3. Observation models: (multivariate) Gaussian

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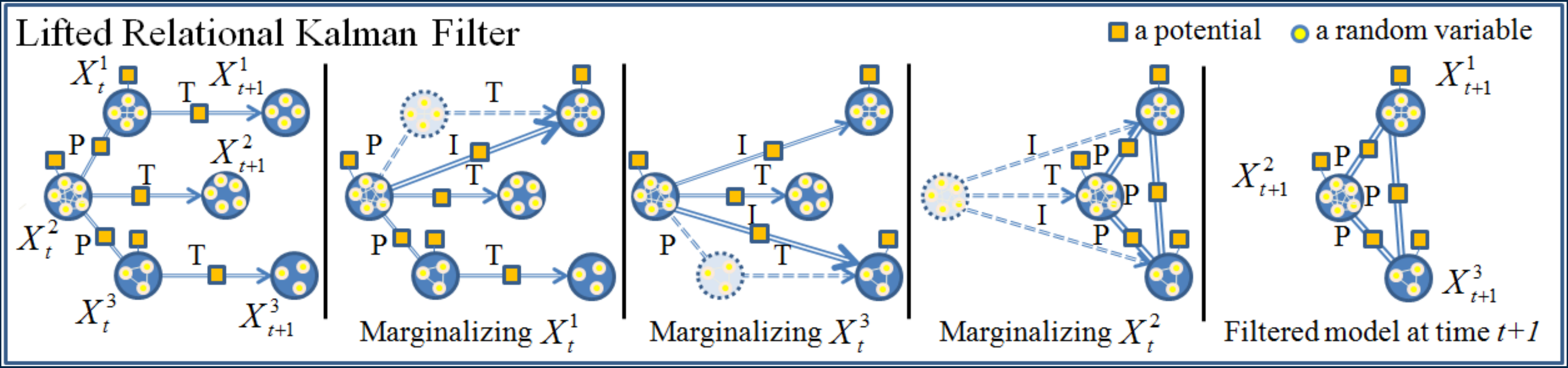
- The KF is of widespread use:
  - Robotics (localization, SLAM (Simultaneous Localization And Mapping)
  - Environmental engineering (weather forecasting)
  - Econometrics (market forecasting)
  - Tracking (object, vehicles, targets, etc.)

Lifted Relational Kalman Filtering (LRKF)

- The Lifted Relational Kalman Filter (LRKF) is a lifted KF. LRKF enables to scale exact KF from 1000 variables to 1,000,000,000 variables.



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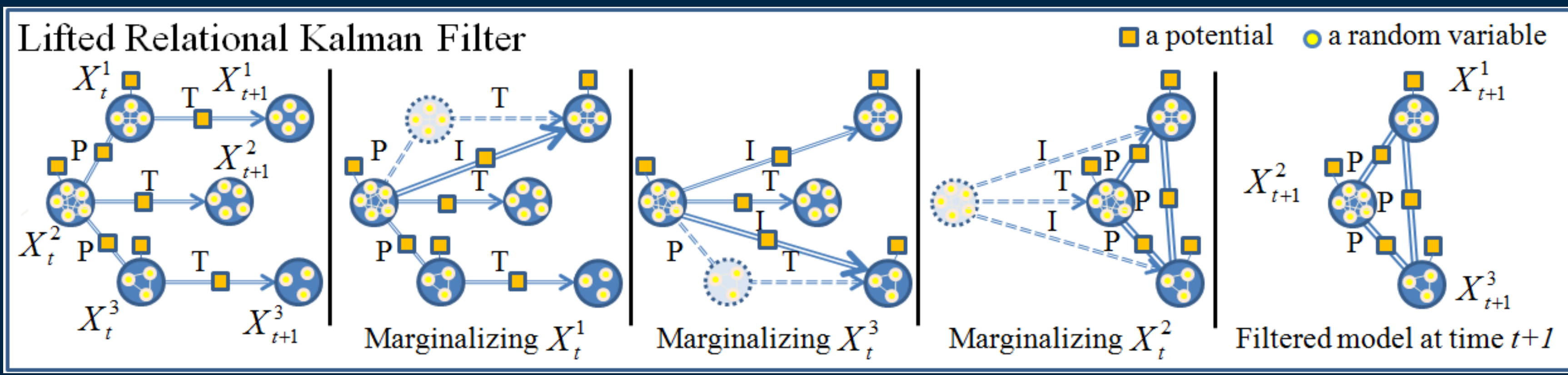

- Key Insights
  - After a filtering step, the pair-wise representation is maintained (product of potentials over two variables each).
  - Two state variables continue to have the same variance and covariances even when individual observations are made.

Outline of the talk

- Relational Gaussian Models (RGMs)
- Problem Definition: Filtering with RGMs
- Lifted Relational Kalman Filtering
- Computational Complexity
- Experimental Results
- Conclusions

Relational Gaussian Models (RGMs)

$$\exp(-(X_t-\mu)\Sigma^{-1}(X_t-\mu)^T)$$

I. Relational Pair-wise Models + Means

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I. Relational Pair-wise Models + Means

- X_t is a disjoint union of m subsets, $X_{t,i}$ (e.g. atoms).

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- Any two state variables in an atom have the same variance and covariances

$$\forall x,x' \in X_{t,i}\ \sigma_x^2 = \sigma_{x'}^2, \quad \forall x,x' \in X_{t,i} \forall y \in X_t \sigma_{x,y} = \sigma_{x',y}$$

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- Any multivariate Gaussian of X_t can be represented as a product of pair-wise potentials, i.e., quadratic exponentials.

$$P(X_t)\propto\prod_{i,j}\prod_{\substack{x\in X_{t,i}\\ y\in X_{t,j}}}\exp\left(-\frac{(x-\beta_{RPM_{i,j}}y-\mu_{RPM_{i,j}})^2}{2\cdot\sigma^2_{RPM_{i,j}}}\right)\prod_{x\in X_t}\exp\left(-\frac{(x-\mu_x)^2}{2\cdot\sigma_x^2}\right)$$

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$$P(X_t) \propto \prod_{i,j} \prod_{\substack{x \in X_{t,i} \\ y \in X_{y,j}}} \phi_{PRM_{i,j}}(x,y) \prod_{x \in X_t} \phi_\mu(x)$$

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Atoms Individuals

Relational Gaussian Models (RGMs)

II. Relational Transition Models

III. Relational Observation Models

Relational Gaussian Models (RGMs)

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$$\forall x \in X_{t,i}, y \in X_{t+1,j} \quad \phi_{RTM_{i,j}}(y_{t+1} | x_t) \propto \exp\left(-\frac{\left(y_{t+1} - \beta_{RTM_{i,j}} x_t\right)^2}{2 \cdot \sigma^2_{RTM_{i,j}}} \right)$$

User inputs are omitted

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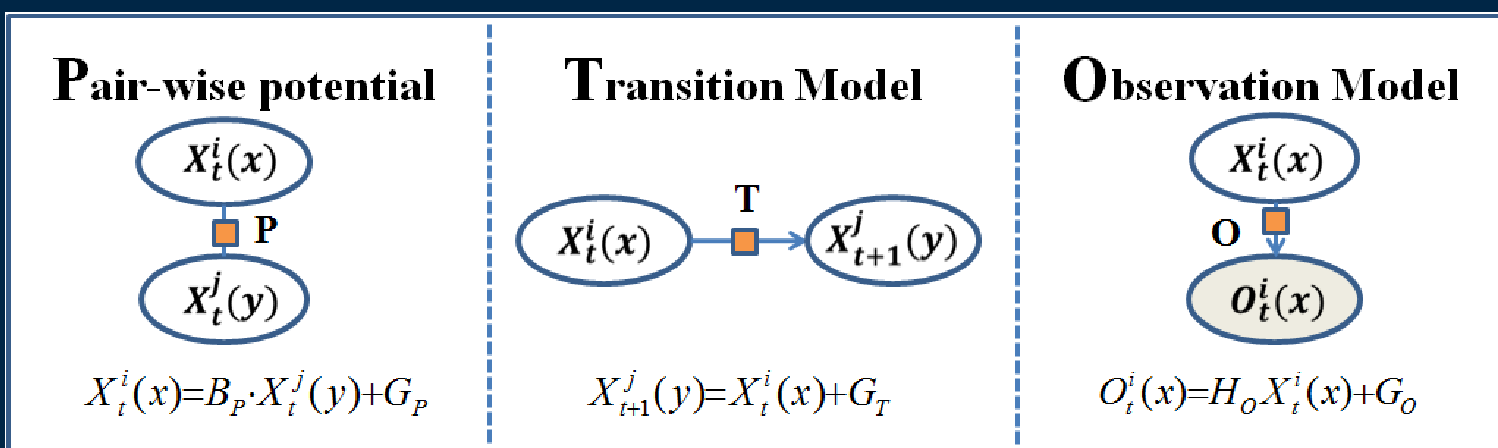
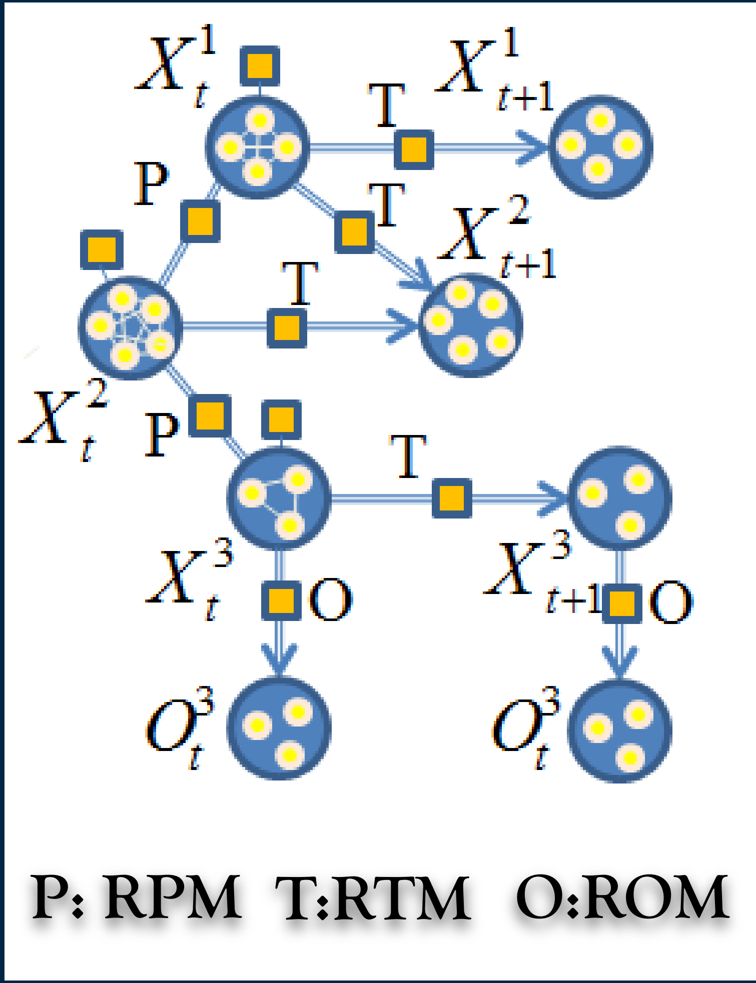


Relational Gaussian Models (RGMs)

- Putting things together (O_{t+1} is a set of observations)



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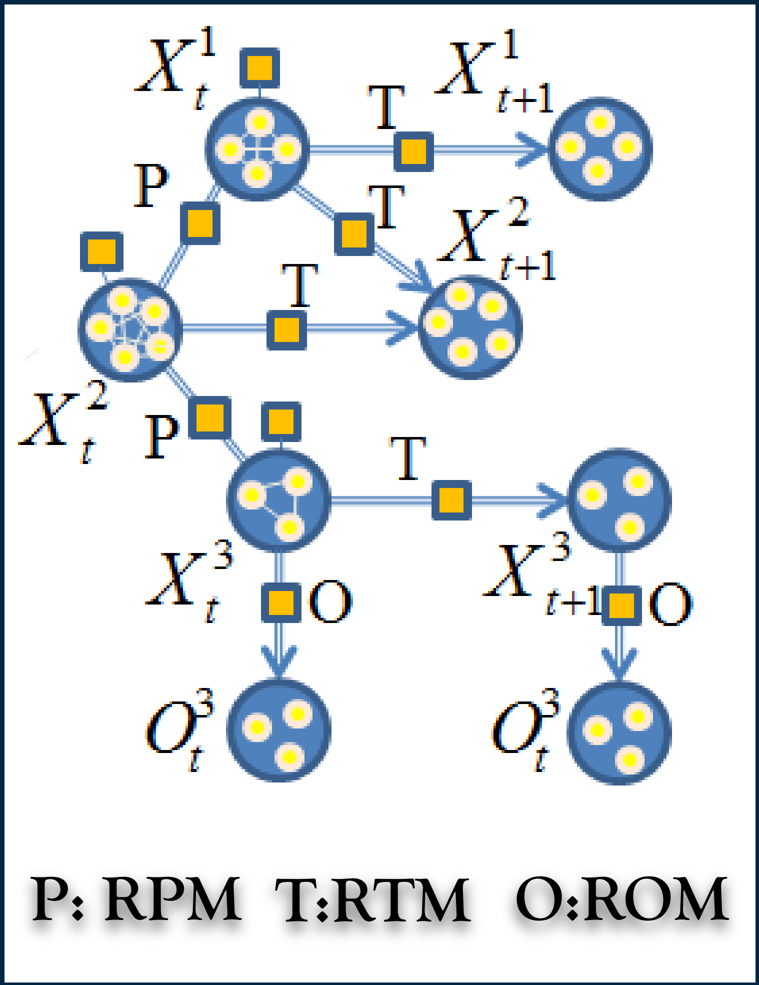

$$P(X_t, X_{t+1} | O_{t+1})$$

$$\propto \prod_{i,j} \prod_{\substack{x \in X_{t,i} \\ y \in X_{t,j}}} \phi_{RPM_{i,j}}(x,y) \cdot \prod_{x \in X_t} \phi_{\mu}(x) \cdot$$

$$\prod_{i,j} \prod_{\substack{x \in X_{t,i} \\ y \in X_{t,j}}} \phi_{RTM_{i,j}}(y_{t+1} | x_t) \cdot$$

$$\prod_{i} \prod_{\substack{o_x \in O_i \\ x \in X_{t+1}}} \phi_{ROM_i}(o_i | x)$$

Lifted Relational Kalman Filtering
Maintaining Pair-wise Potentials

- Filtering is inference with RGMs:
  - Marginalize all state variables of timestep t. $\int P(X_t, X_{t+1} | O) dX_t$
  - Marginalize a variable $x \in X_t$ ($X'_t = X_t \backslash x$)
  - Marginalization preserves pair-wise potentials.
  - Continue to marginalize all remaining variables.

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$$\int\int \exp\left(-Ax^2 + 2Bx + C \right) dx\, dX_t' = \int \frac{\sqrt{\pi}}{\sqrt{A}} \exp\left(B^2 / A + C \right) dX_t'$$

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Constant | Sum of variables

$$B = \sum_i \left(\sum_{y\in X'_{t,i}} c_{t,i} y + \sum_{y\in X_{t+1,i}} c_{t+1,i} y \right) + \epsilon$$

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Constant **Sum of variables**

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  - Marginalization preserves pair-wise potentials.

$$B^2 = \sum_i \left(c^2_{t,i} \sum_{y \in X'_{t,i}} y^2 + c^2_{t+1,i} \sum_{y \in X_{t+1,i}} y^2 \right) + 2 \sum_{i,j} \left(c_{t,i} c_{t,j} \sum_{\substack{y \in X'_{t,i} \\ y' \in X'_{t,j}}} yy' + c_{t,i} c_{t+1,j} \sum_{\substack{y \in X'_{t,i} \\ y' \in X'_{t+1,j}}} yy' + c_{t+1,i} c_{t+1,j} \sum_{\substack{y \in X'_{t+1,i} \\ y' \in X'_{t+1,j}}} yy' \right) + \alpha$$

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$$\int\int \exp\left(\underbrace{\boxed{-A}}_{\text{Constant}}\, x^2 + \underbrace{\boxed{2B}}_{\text{Sum of variables}}\, x + C\right) dx\, dX'_t = \int \frac{\sqrt{\pi}}{\sqrt{A}} \exp\left(B^2/A + C\right) dX'_t$$

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  - Marginalization preserves pair-wise potentials.

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  - Continue to marginalize all remaining variables.

Lifted Relational Kalman Filtering

Handling Individual Observations

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Lifted Relational Kalman Filtering

Splitting atoms (groups) given observations

- Splitting cases

No split

Split

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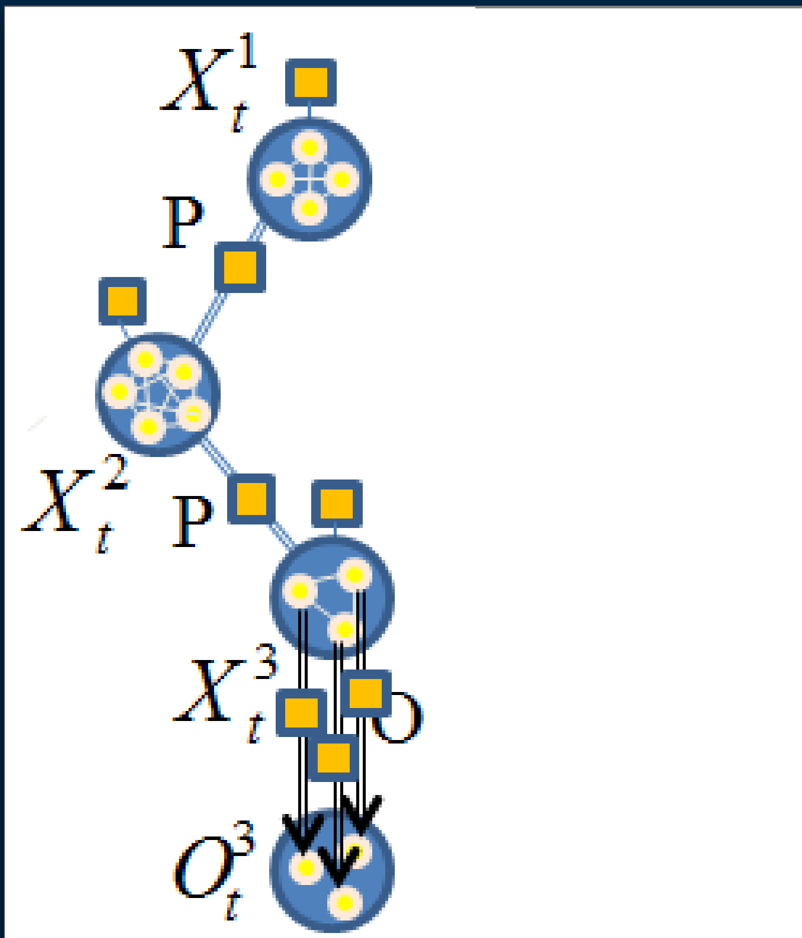

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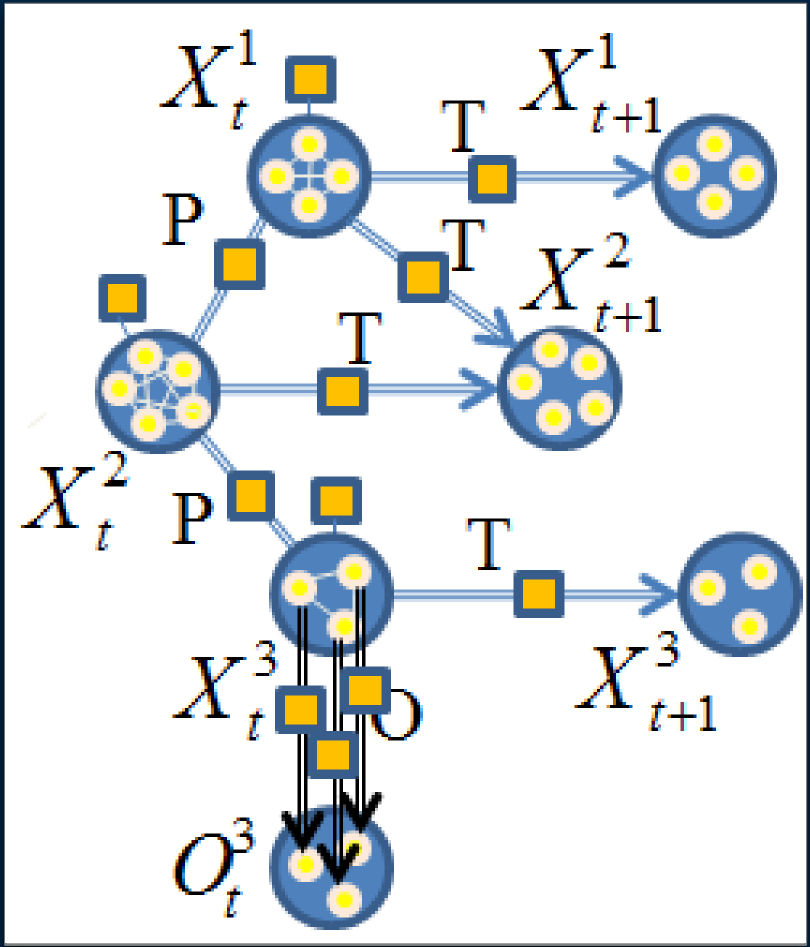

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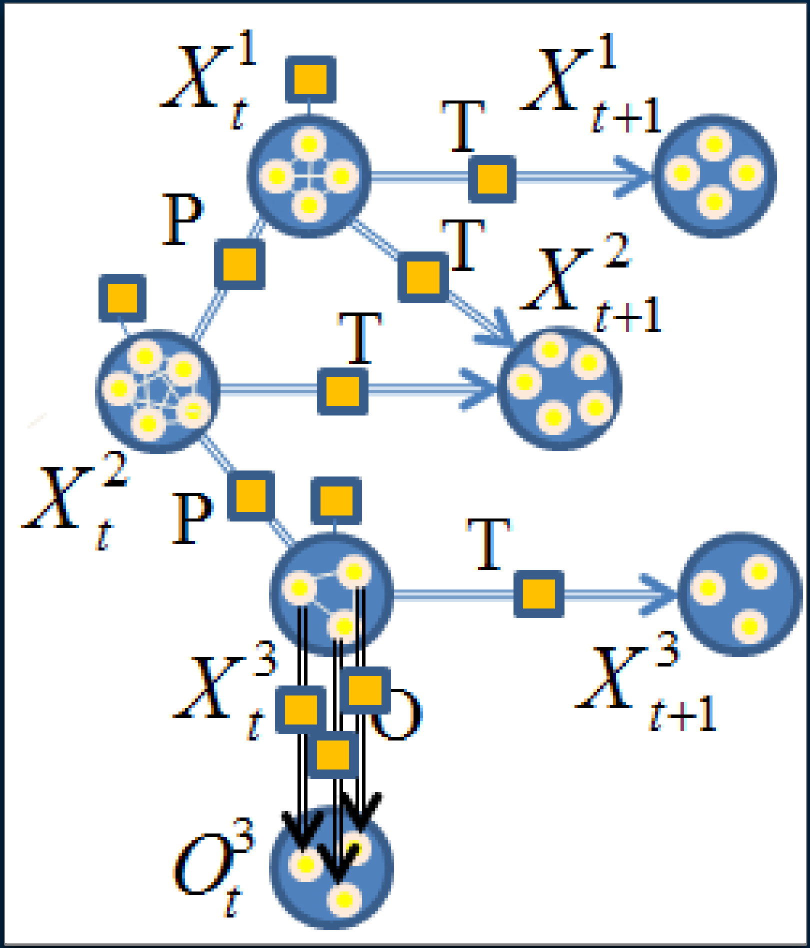

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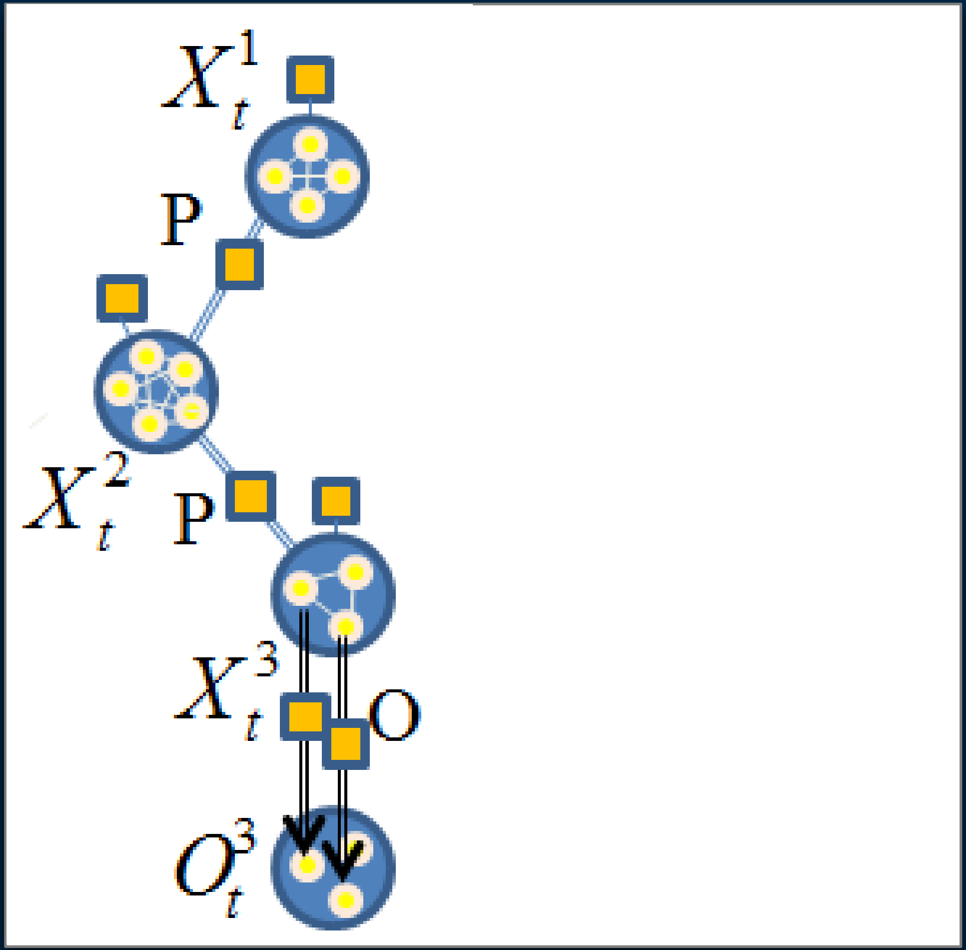

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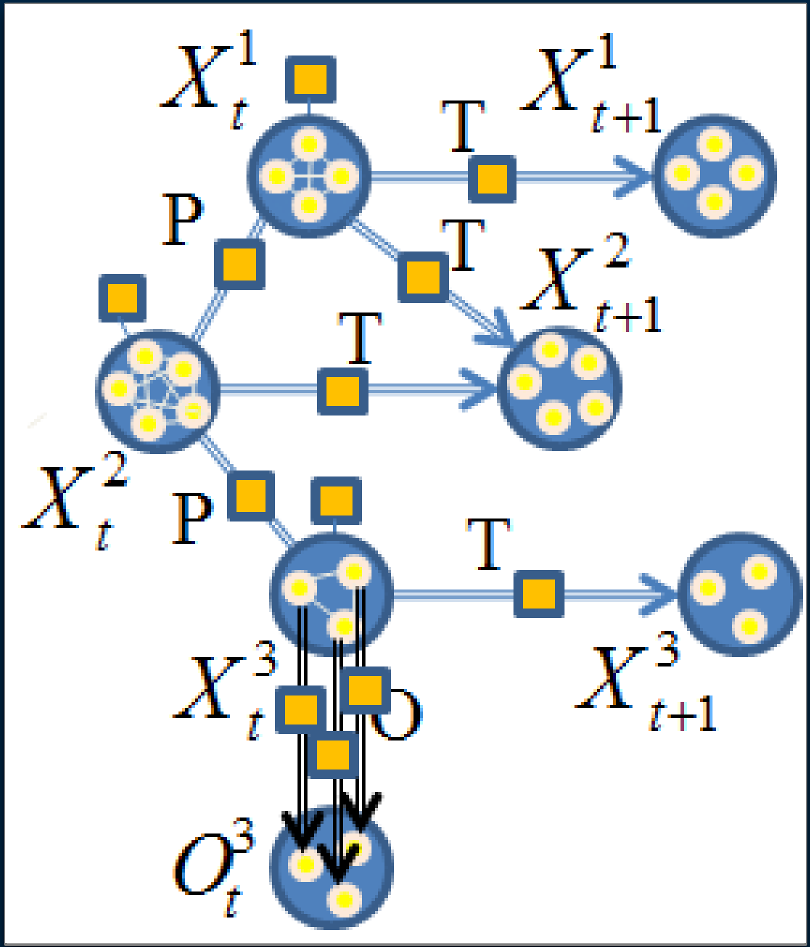

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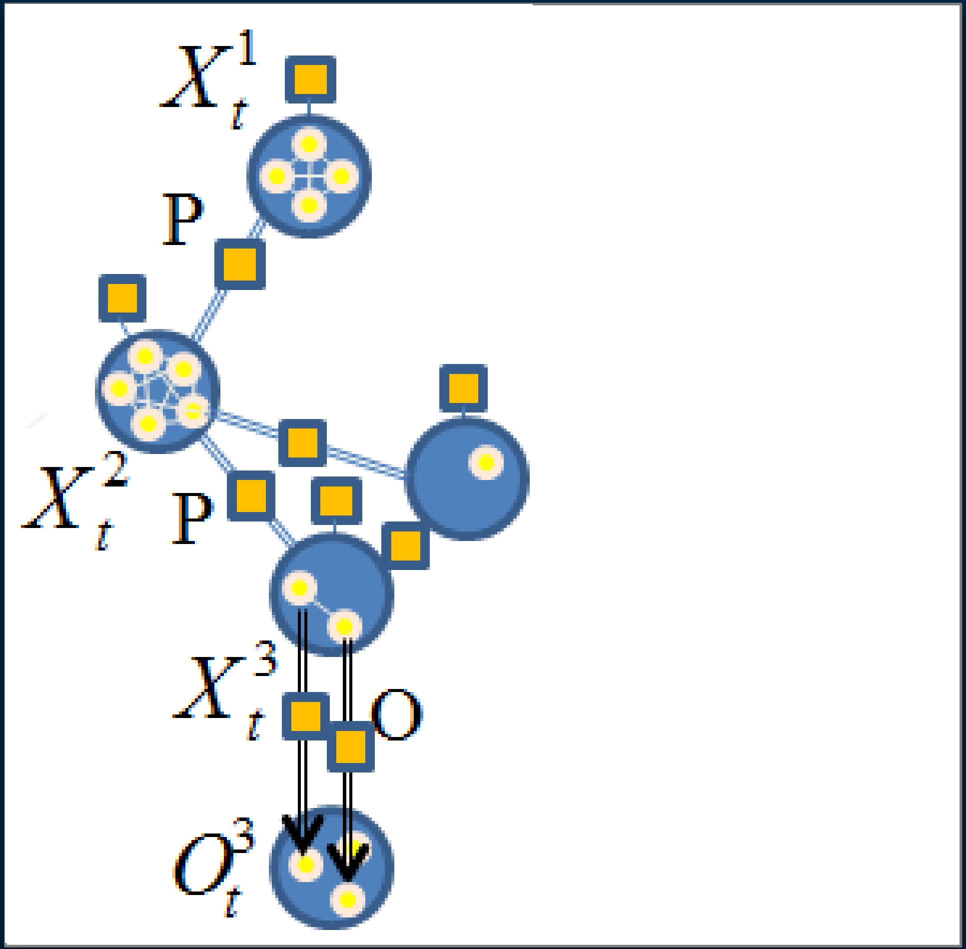

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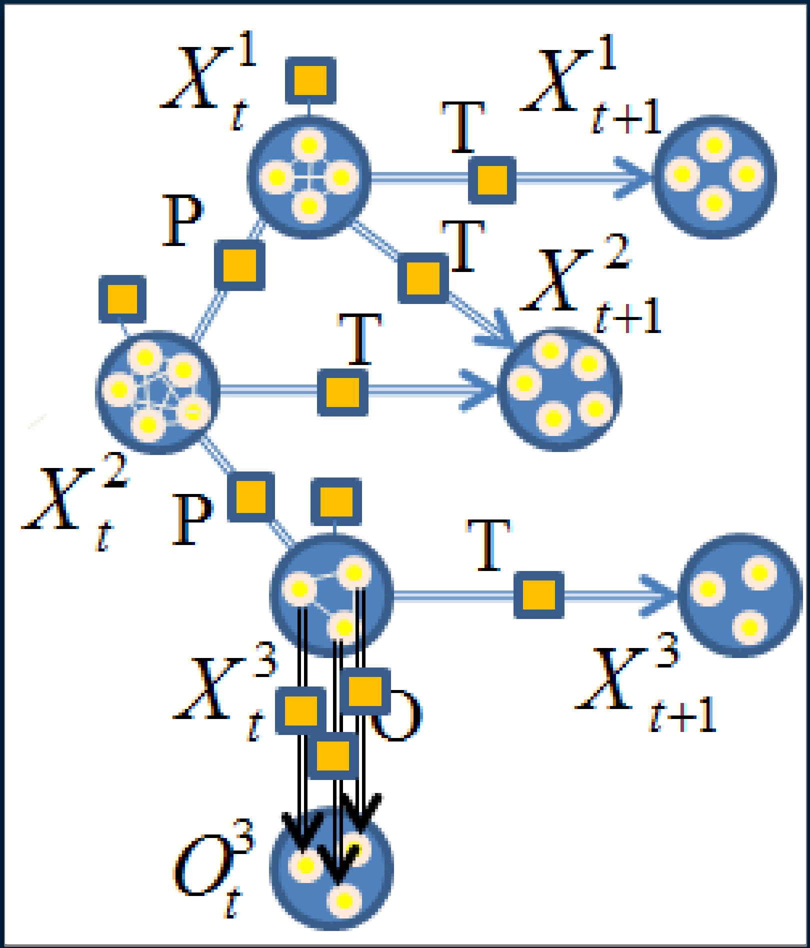

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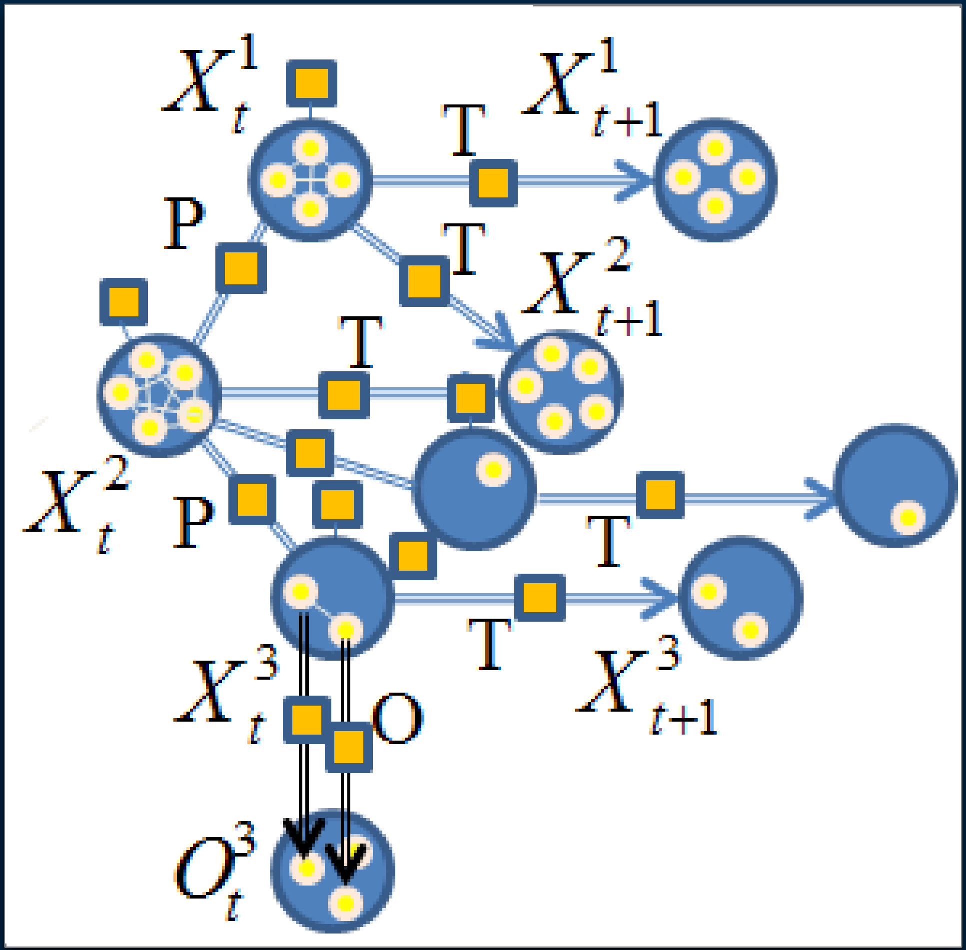


Split

Computational Complexity of 'LRKF'

This algorithm (LRKF): $O(n \cdot m^2)$ $n \gg m$

Ground KF: $O(n^3)$

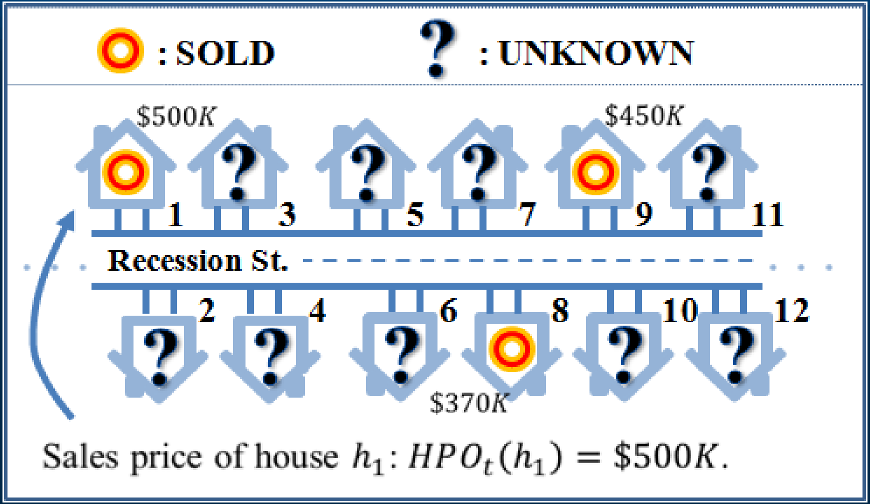
n: the number of state random variables

m: the size of the induced partition (number of atoms in the RGM)

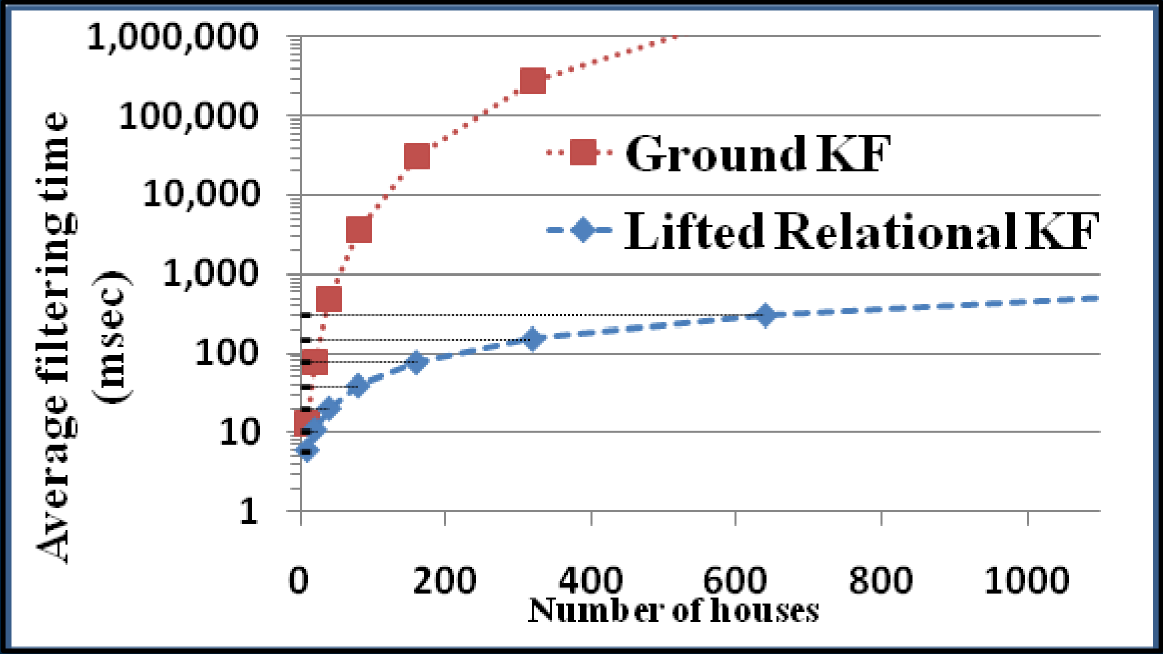
- **Therefore, LRKF enables exact Kalman filtering of 1,000,000,000 state variables (in contrast to 1000 variables with the original KF).**

Experimental Results

- Experiments setup
  - Given: housing market example.
  - Observations for:
    - Housing market index,
    - Sales prices for a set of houses



- Goal: Estimate the price of each house (mean and variance)

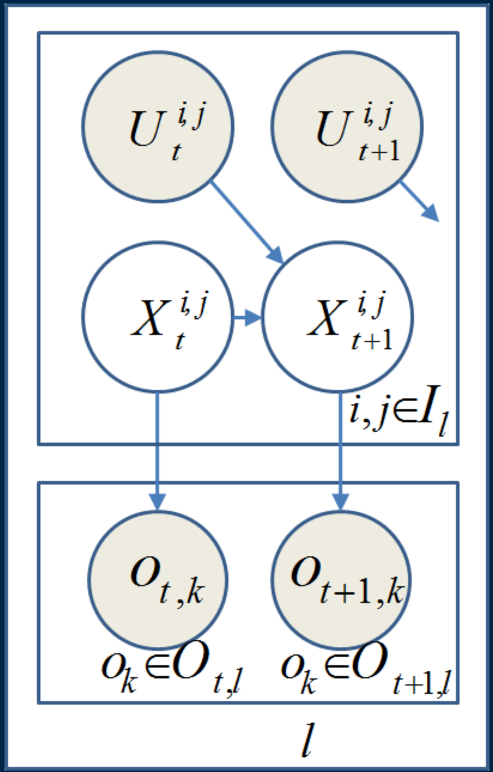


Experimental Results

- An Application to online Social Networks
  - Our LRKF can extend an inference problem [Xian, Neville and Rogati, WWW, 2011] in social networks into a filtering problem.
    - $U_{i,j}$: profile similarity feature vectors
    - $X_{i,j}$: the relationship strength variables
    - O_k: interaction observations

$$\prod_{o_{t+1}^k \in O_{t+1}} \phi_{ROM_k}(o_{t+1}^k, X_{t+1}(i,j)) = \prod_{o_{t+1}^k \in O_{t+1}} \exp\left(-\frac{(o_{t+1}^k - \beta_{O,k} X_{t+1}(i,j))^2}{2\sigma_{O,k}^2}\right)$$

$$\phi_{RTM}(X_{t+1}(i,j), X_t(i,j), \mathbf{U}_t(i,j)) = \exp\left(-\frac{(X_{t+1}(i,j) - \mathbf{w}\cdot\mathbf{U}_t(i,j) - \beta_T X_t(i,j))^2}{2\sigma_T^2}\right)$$



Conclusions

- We present a lifted inference algorithm that enables linear time exact Kalman filtering **with 1,000,000,000 state variables** in contrast to the traditional **KF which can only handle1000 state variables**.
- We show that a lifted inference is still possible even when individual observations are made for all random variables.

References

- Kalman Filters
  - Sparse matrix: Fast Kalman Filter [Lange, 2001], Bayes Tree [Kaess, 2010]
  - Sampling: Ensemble Kalman Filter [Evensen, 1994]
- Solving linear Gaussian as an inference problem
  - Gaussian Markov Random Fields [Rue and Held,2005] and Directed Gaussian Models [Cowell,1998]
- Lifted inferences for relational models:
  - Discrete: [Poole,2003], [Braz,Amir and Roth,2005] and [Milch et al.,2008] ,[Singla and Domingos, 2008] ...
  - Continuous: [Wang and Domingos, 2008],[Choi,Hill and Amir, 2010]

Thank you

Questions or suggestions?

please contact jaesik@illinois.edu